1. Find the values of all six trigonometric functions at $\theta = \frac{11\pi}{6}$ (that is, find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\tan\theta$.) *Completely simplify all values.*

   Solution. The angle $\theta = \frac{11\pi}{6}$ is in QIV with a reference angle of $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. So

   $$\sin\frac{11\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2} \qquad \csc\frac{11\pi}{6} = -2$$

   $$\cos\frac{11\pi}{6} = +\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sec\frac{11\pi}{6} = +\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

   $$\tan\frac{11\pi}{6} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \qquad \cot\frac{11\pi}{6} = -\sqrt{3}$$

   Notes:

   If you don't simplify your answers completely as done above, you will not receive full credit.

2. Find the values of all six trigonometric functions at $\theta = \frac{3\pi}{2}$. *Completely simplify all values.*

Solution. The angle $\theta = \frac{3\pi}{2}$ is a special case. You should know $\sin\frac{3\pi}{2} = -1$ and $\cos\frac{3\pi}{2} = 0$. Thus

$$\sin\frac{3\pi}{2} = -1 \qquad \csc\frac{3\pi}{2} = -1$$

$$\cos\frac{3\pi}{2} = 0 \qquad \sec\frac{3\pi}{2} = \frac{1}{0} = \text{undefined}$$

$$\tan\frac{3\pi}{2} = \frac{\sin\frac{3\pi}{2}}{\cos\frac{3\pi}{2}} = \frac{-1}{0} = \text{undefined} \qquad \cot\frac{3\pi}{2} = \frac{0}{-1} = 0$$

3. Solve each of the following equations for θ in the interval $[0, 2\pi]$.

(a) $\csc 2\theta = -2$.

Solution.

$$\begin{aligned}
\csc 2\theta &= -2 \\
\sin 2\theta &= -\frac{1}{2} \\
2\theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \\
\theta &= \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}
\end{aligned}$$

(b) $4\sin\theta - 2\csc\theta = 0$

Solution.

$$\begin{aligned}
4\sin\theta - 2\csc\theta &= 0 \\
4\sin\theta - 2\cdot\frac{1}{\sin\theta} &= 0 && (*) \\
4\sin^2\theta - 2 &= 0 && (**) \\
4\sin^2\theta &= 2 \\
\sin^2\theta &= \frac{1}{2} \\
\sin\theta &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= \frac{1}{\sqrt{2}} && \text{or} && \sin\theta = -\frac{1}{\sqrt{2}} && (***) \\
\theta &= \frac{\pi}{4}, \frac{3\pi}{4} && \text{or} && \theta = \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

Notes:

- In $(*)$, we used the identity $\csc\theta = \dfrac{1}{\sin\theta}$.
- In $(**)$, we multiplied through by the LCD $= \sin\theta$.
- In $(***)$, in both cases, the reference angle is $\dfrac{\pi}{4}$.

(c) $3\cot^2\theta - 1 = 0$

Solution.

$$\begin{aligned} 3\cot^2\theta - 1 &= 0 \\ 3\cot^2\theta &= 1 \\ \cot^2\theta &= \frac{1}{\sqrt{3}} \\ \cot\theta &= \pm\frac{1}{\sqrt{3}} \\ \tan\theta &= \pm\sqrt{3} \end{aligned}$$

$$\tan\theta = \sqrt{3} \qquad \text{or} \qquad \tan\theta = -\sqrt{3} \qquad (*)$$

$$\theta = \frac{\pi}{3}, \frac{4\pi}{3} \qquad \text{or} \qquad \theta = \frac{2\pi}{3}, \frac{5\pi}{3}$$

Notes:

- In $(*)$, in both cases, the reference angle is $\frac{\pi}{3}$.

(d) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution.

Use the identity $\cos^2\theta = 1 - \sin^2\theta$ (which comes from $\sin^2\theta + \cos^2\theta = 1$) so that the equation has only $\sin\theta$'s to deal with.

$$\begin{aligned}
2\cos^2\theta + \sin\theta - 1 &= 0 \\
2(1 - \sin^2\theta) + \sin\theta - 1 &= 0 \\
2 - 2\sin^2\theta + \sin\theta - 1 &= 0 \\
-2\sin^2\theta + \sin\theta + 1 &= 0 \\
2\sin^2\theta - \sin\theta - 1 &= 0 \\
(2\sin\theta + 1)(\sin\theta - 1) &= 0
\end{aligned}$$

$$\begin{aligned}
2\sin\theta + 1 &= 0 &\qquad \text{or} \qquad& \sin\theta - 1 = 0 \\
\sin\theta &= -\frac{1}{2} &\qquad \text{or} \qquad& \sin\theta = 1 \\
\theta &= \frac{7\pi}{6}, \frac{11\pi}{6} &\qquad \text{or} \qquad& \theta = \frac{\pi}{2}
\end{aligned}$$

(e) $\sin\theta - \cos\theta = 1$ (this one is tricky)

Solution. To relate $\sin\theta$ and $\cos\theta$, we'll use the identity $\sin^2\theta + \cos^2\theta = 1$. However before we can use that identity, we need to isolate $\sin\theta$ (or $\cos\theta$) and square both sides.

$$\begin{aligned}\sin\theta - \cos\theta &= 1\\ \sin\theta &= \cos\theta + 1\\ (\sin\theta)^2 &= (\cos\theta+1)^2 \qquad (*)\\ \sin^2\theta &= \cos^2\theta + 2\cos\theta + 1\\ 1-\cos^2\theta &= \cos^2\theta + 2\cos\theta + 1\\ 2\cos^2\theta + 2\cos\theta &= 0\\ 2\cos\theta(\cos\theta+1) &= 0\end{aligned}$$

$$\begin{array}{lcl} 2\cos\theta = 0 & \text{or} & \cos\theta + 1 = 0\\ \cos\theta = 0 & \text{or} & \cos\theta = -1\\ \theta = \frac{\pi}{2}, \frac{3\pi}{2} & \text{or} & \theta = \pi\end{array}$$

In $(*)$, we squared both sides which may introduce extraneous solutions. So we have to check our solutions:

for $\theta = \frac{\pi}{2}$,
$$\begin{aligned}\sin\theta - \cos\theta &\stackrel{?}{=} 1\\ \sin\frac{\pi}{2} - \cos\frac{\pi}{2} &\stackrel{?}{=} 1\\ 1 - 0 &\stackrel{?}{=} 1\\ 1 &= 1\end{aligned}$$

for $\theta = \frac{3\pi}{2}$,
$$\begin{aligned}\sin\theta - \cos\theta &\stackrel{?}{=} 1\\ \sin\frac{3\pi}{2} - \cos\frac{3\pi}{2} &\stackrel{?}{=} 1\\ -1 - 0 &\stackrel{?}{=} 1\\ -1 &\neq 1\end{aligned}$$

for $\theta = \pi$,
$$\begin{aligned}\sin\theta - \cos\theta &\stackrel{?}{=} 1\\ \sin\pi - \cos\pi &\stackrel{?}{=} 1\\ 0 - (-1) &\stackrel{?}{=} 1\\ 1 &= 1\end{aligned}$$

So the only solutions are $\theta = \frac{\pi}{2}, \pi$.

4. Given $\sin\theta = -\frac{7}{8}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$, find the values of the other five trigonometric functions.

Solution.

$$\sin\theta = -\frac{7}{8} \qquad \csc\theta = -\frac{8}{7}$$

$$\cos\theta = -\frac{\sqrt{15}}{9} \qquad \sec\theta = -\frac{9}{\sqrt{15}} = -\frac{9\sqrt{15}}{15} = -\frac{3\sqrt{15}}{5}$$

$$\tan\theta = +\frac{7}{\sqrt{15}} = \frac{7\sqrt{15}}{15} \qquad \cot\theta = +\frac{\sqrt{15}}{7}$$

Notes:

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5. Given $\cot\theta = -3$ and $\frac{3\pi}{2} \le \theta \le 2\pi$, find the values of the other five trigonometric functions.

Solution.

$$\sin\theta = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10} \qquad \csc\theta = -\sqrt{10}$$

$$\cos\theta = +\frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10} \qquad \sec\theta = +\frac{\sqrt{10}}{3}$$

$$\tan\theta = -\frac{1}{3} \qquad \cot\theta = -3$$

Notes:

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