INVARIANCE OF THE WILANSKY PROPERTY

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ABSTRACT. The Wilansky property was originally formulated as a property of FK spaces in 1987 by both G. Bennett [1] and W. Stadler [12]. In 1991, A. K. Snyder and G. Stoudt [10] identi¯ed the Wilansky property as a basis property for a Banach space. In the present work, we study the invariance of this property with respect to the coe±cient functionals associated with a basis. We characterize sectional boundedness of biorthogonal systems in Banach spaces in terms of the closed linear span of its functionals. This characterization is then employed to prove that the Wilansky property, as de¯ned for a basis, is an invariant of the closed span of its coe±cient functionals.

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1. Introduction

It is known that for X and Y FK containing '; if X µ Y; then Y^f µ X^f: In [11], Snyder and Wilansky proved that if ' is dense in X and Y is FK, then X ½ Y i® Y^f ½ X^f. In [14], Wilansky noted that this is not true for ¯-duals and cited an example. However, he posed the following question: If ' is dense in the FK space X where X µ c_0 and $X^{\bar{}}$ = $c_0^{\bar{}}$ = ℓ_1, must X = c_0? Independently, G. Bennett [1] and W. Stadler [12] answered the question a±rmatively in 1987 and showed it is not necessarily true if c_0 is replaced by any sequence space. Both went on to generalize this for any FK space dense in another with corresponding ¯-duals. Bennett de¯ned what is known as the Wilansky property, which will be abbreviated (W), as follows.

DEFINITION 1.1. An FK space Z is said to have the Wilansky property if X = Z whenever X is an FK space dense in Z such that $Z^{\bar{}}$ = $X^{\bar{}}$.

For example, c_0, $\ell_p(1 < p < 1)$, c, and cs have (W), whereas ℓ_1, ℓ_1, !, bv, and bs fail (W). It was shown in [5] that the ¯-dual may be replaced by the °-dual for an equivalent de¯nition.

Bennett [1], Noll, and Stadler [6] established a connection between the Wilansky property and the closed span of the coe±cient functionals, $[\phi_n]$; in a BK space with the basis $\{\pm^n\}$: In the papers that introduced (W), both Bennett [1] and Stadler [12] proved that if Z is BK with basis $\{\pm^n\}$ and $Z^0 = [\phi_n]_{Z^0}$ then Z has (W). In 1989 D. Noll and W. Stadler [6] improved this statement by showing if Z is BK with basis $\{\pm^n\}$ and $[\phi_n]_{Z^0}$ has ¯nite codimension in Z^0, then Z has (W). Relating (W) directly to a general basis was a natural area for further study.

In 1991 A. Snyder and G. Stoudt [10] gave the following de¯nition for a basis to have the Wilansky property, also abbreviated (W).

DEFINITION 1.2. A basis $\{z_n\}$ for a Banach space Z has the Wilansky property i® X has ¯nite codimension in Z whenever X is BZ with the property that Z is the smallest BZ space between X and Z in which $\{z_n\}$ is basic.

As examples, the standard bases for both c_0 and $\ell_p (1 < p < \infty)$ have (W) whereas the standard basis for ℓ_1 fails (W). It is clear that (W) is invariant under equivalent bases since basic sequences are preserved under isomorphisms. One might conjecture that (W) is an invariant of the space, but Snyder [9] shows that $cs(c_0)$ is isomorphic to c_0, and the standard basis for c_0 has (W) whereas the basis he constructs for $cs(c_0)$ fails (W). With the connection between (W) and $[\phi_n]$ established by Bennett, Stadler and Noll, one might also conjecture that (W); as de¯ned for a basis, is an invariant of the closed span of the coe±cient functionals. This is shown in this paper as Theorem 5.8 in the following form.

THEOREM. Let $\{x_n\}$ and $\{y_n\}$ be bases for the Banach space Z with coe±cient functionals $\{f_n\}$ and $\{g_n\}$, respectively. If $\{x_n\}$ has (W) and $[f_n] \mu [g_n]$, then $\{y_n\}$ has (W).

This theorem is proven in [9] with the use of strictly cosingular operators. As an alternative approach to the proof, we study biorthogonal systems and their connection to the Wilansky property.

In Section 3, we give a basis result which follows when the closed spans of coe±cient functionals are included as in the theorem above. Several properties which follow from sectional boundedness of a biorthogonal system are presented in Section 4. In the main theorem of this section, we characterize the sectional boundedness of a biorthogonal system in terms of the closed span of its functionals. This characterization is then employed to prove the main theorem in Section 5 which veri¯es the invariance property of (W).

2. Notation and Terminology

The material presented in this section can be found in [14] and [8].

A sequence space is a linear subspace of ω, the space of all scalar sequences. ω is a Fréchet space with the weak topology by the family of coordinate functionals $\{\varphi_k : k = 1, 2, \ldots\}$, which are given by $\varphi_k(x) = x_k$ for all $x \in \omega$. An FK space is a Fréchet subspace of ω with the property that the restriction of each coordinate functional to the subspace is continuous. A BK space is the Banach equivalent. Sequences in Banach spaces will be denoted with subscripts while those in BK spaces will bear superscripts with subscripts to index the coordinates. The linear span of $\{\delta^n : n = 1, 2, \ldots\}$, where $\delta^n_n = 1$ and $\delta^n_k = 0$ for $k \neq n$, will be denoted by φ. X^0 will denote the space of continuous linear functionals on X.

The $^\circ$-dual and $^-$-dual of a sequence space X are given by

$$X^\circ = \left\{ u \in \omega : \|u x\|_{bs} = \sup_n \left| \sum_{k=1}^{n} u_k x_k \right| < 1 \text{ for all } x \in X \right\} \text{ and}$$

$$X^- = \left\{ u \in \omega : \sum u_n x_n \text{ converges under } \|\cdot\|_{bs} \text{ for all } x \in X \right\}.$$

The functional dual for a sequence space X containing φ is given by $X^f = \{f(\delta^n) : f \in X^0\}$.

A sequence $\{x_n\}$ in a Banach space X is a (Schauder) basis for X if for each $x \in X$ there exists a unique $\alpha \in \omega$ such that $x = \sum \alpha_k x_k$, the series converging in norm. A sequence $\{x_n\}$ in a Banach space is basic if it is a basis for its closed linear span, $\overline{\operatorname{span}\{x_n : n = 1, 2, \ldots\}}$, which will be denoted as $[x_n]$. With $\{x_n\}$ as a basis for X, there exists a sequence $\{f_k\} \subset X^0$, the associated sequence of coefficient functionals, such that for each $x \in X$ we have the representation $x = \sum f_k(x) x_k$.

An AK space is a space for which $\{\delta^n\}$ is a basis. If X is BK AK, then all the duals mentioned are isomorphic, and X^0 may be considered as a sequence space. An FK space X containing φ is AD, or sectionally dense, if φ is dense in X. If X is FK, then $X^0 = X^f$ in the sense of the isomorphism $q : X^0 \to X^f$ given by $q(f) = \{f(\delta^n)\}$ if X is AD. An FK space X containing φ is AB, or sectionally bounded if $\left\{ \sum_{k=1}^{n} \varphi_k(x) \delta^k \right\}$ is bounded in X for all $x \in X$. Analogous properties for biorthogonal systems in Banach spaces are defined as follows.

For a Banach space X with $\{x_n\} \subset X$ and $\{f_n\} \subset X^0$, the pair $\{x_n; f_n\}$ is a biorthogonal system for X if $f_n(x_k) = \delta^n_k$. The system is complete if $[x_n] = X$. The system is total if $\operatorname{span}\{f_n : n = 1, 2, \ldots\}$ is w^*-dense in X^0, that is $[f_n]_{w^*} = X^0$. Any AK space has $\{\delta^n; \varphi_n\}$ as a

complete, total biorthogonal system. In general, if $\{x_n\}$ is a basis for a Banach space X with coe±cient functionals $\{f_n\}$, then $\{x_n; f_n\}$ is a complete, total biorthogonal system in X with $\{f_n\}$ forming a weak$^{\alpha}$ basis for X^0:

The biorthogonal system is sectionally bounded, or AB, i® $\left\{\sum_{k=1} f_k(x)x_k\right\}$ is bounded in X for each $x \in X$: Notice that $\{x_n; f_n\}$ is AB i® the nth section maps $P_n : X \to X$ given by $P_n x = \sum_{k=1} f_k(x)x_k$ are bounded for each $x \in X$. Therefore, $\{x_n; f_n\}$ is AB i® $\{P_n\}$ is bounded.

When $\{x_n\}$ is a basis for for a Banach space X with coe±cient functionals $\{f_n\}$, then the map $x \to \{f_k(x)\}$ identi¯es X with the BK AK space $T(X) = \{\{f_k(x)\} : x \in X\}$ where each x_n is identi¯ed with $\pm^n$ and each f_n with φ_n. Working with the BK AK sequence space representation rather than the Banach space with its basis will enable us to apply results from classical sequence space theory.

When X is a subspace of the Banach space Z with a di®erent norm, then $\|x\|_X$ denotes the norm of x in X and $\|x\|_Z$ denotes the norm of x in Z. Similarly, for $S \subseteq X$, $[S]_X$ denotes the closed span of S in X and $[S]_Z$ the closed span in Z. Note that if $S \subseteq X \subseteq Z$, then $[S]_X \subseteq [S]_Z$. As closed linear spans are often referenced, X_0 will be used as a substitute for $[x_n]_X$ when there is no ambiguity. Thus, for sequence spaces X with $\varphi \subseteq X$, X_0 will denote $[\pm^n]_X$ as usual.

3. A Basis Result

Some basic functional analysis facts are needed for the theorem which follows.

FACT 3.1. If X is a Banach space and $T \in B(X; X)$ with $\|T\| < 1$; then $I - T$ is invertible.

Two bases, $\{x_n\}$ and $\{y_n\}$ for the Banach spaces X and Y, respectively, are equivalent i® the convergence of $\sum \alpha_k x_k$ in X is equivalent to the convergence of $\sum \alpha_k y_k$ in Y. It is known that $\{x_n\}$ and $\{y_n\}$ are equivalent i® there is an isomorphism between X and Y that maps each x_n to y_n. As shown in the following theorem, given two bases for a Banach space, if the closed span of one set of coe±cient functionals is contained in the other, then a third basis is produced with nice inclusion properties.

THEOREM 3.2. Let $\{x_n\}$ and $\{y_n\}$ be bases for the Banach space X with coe±cient functionals $\{f_n\}$ and $\{g_n\}$, respectively. If $[f_n] \subseteq [g_n]$; then there exists a basis $\{z_n\}$ for X; with coe±cient functionals $\{h_n\}$;

which is equivalent to $\{x_n\}$ with the properties:

(a) $y_n \in \operatorname{span}\{z_k\}$ for each n
(b) $[h_n] \subset [g_n]$:

Proof. Since $\{x_n\}$ is a basis for X with coefficient functionals $\{f_n\}$, then $y_n = \sum_{k=1}^{\infty} f_k(y_n)x_k$ for each positive integer n. So, for each n, $\|y_n - \sum_{k=1}^{\infty} f_k(y_n)x_k\|$ can be made small by choosing a sufficiently large r. Thus, for each n, r_n can be chosen such that

$$\sum_{n=1}^{\infty} \|g_n\| \left\| y_n - \sum_{k=1}^{\infty} f_k(y_n)x_k \right\| < 1.$$

Let $T : X \to X$ be defined by

$$Tz = \sum_{n=1}^{\infty} g_n(z)\left(y_n - \sum_{k=1}^{\infty} f_k(y_n)x_k \right).$$

T is continuous since

$$\begin{aligned} \|T\| &\le \sup\left\{ \sum_{n=1}^{\infty} \|g_n\| \cdot \|z\| \cdot \left\| y_n - \sum_{k=1}^{r_n} f_k(y_n)x_k \right\| : \|z\| \le 1 \right\} \\ &\le \sum_{n=1}^{\infty} \|g_n\| \cdot \left\| y_n - \sum_{k=1}^{r_n} f_k(y_n)x_k \right\| \\ &< 1 \text{ by choice of the } r_n\text{'s.} \end{aligned}$$

Define $S : X \to X$ by $Sx = x - Tx$, that is, $S = I - T$. By Fact 3.1, S is invertible and we have S as a continuous automorphism of X. Therefore, $\{S^{-1}x_n\}$ is a basis for X equivalent to $\{x_n\}$. Let $z_j = S^{-1}x_j$ for all j. Notice that

$$(S^{*}f_n)z_j = f_n(Sz_j) = f_n(x_j) = \delta_j^n.$$

So, $\{S^{*}f_n\}$ is the sequence of coefficient functionals for the basis $\{z_n\}$.

Let $h_n = S^{*}f_n$ for each positive integer n. To show (a), it suffices to show $Sy_n \in \operatorname{span}\{x_k\}$ for each n. Thus, we have (a) since

$$\begin{aligned} Sy_j &= y_j - \sum_{n=1}^{\infty} g_n(y_j)\left(y_n - \sum_{k=1}^{r_n} f_k(y_n)x_k \right) \\ &= y_j - y_j + \sum_{k=1}^{\infty} f_k(y_j)x_k \quad \text{by biorthogonality of } \{y_n; g_n\} \\ &\in \operatorname{span}\{x_k\}. \end{aligned}$$

To show $[S^0 f_n] = [h_n] \subseteq [g_n]$, $\bar{x}$ j and consider $S^0 f_j - f_j$.

$$\begin{aligned}
(S^0 f_j - f_j)z &= f_j\left(z - \sum_{n=1}^{\infty} g_n(z)\left(y_n - \sum_{k=1}^{n} f_k(y_n)x_k\right)\right) - f_j(z) \\
&= -f_j\left(\sum_{n=1}^{\infty} g_n(z)y_n\right) + f_j\left(\sum_{n=1}^{\infty} g_n(z)\sum_{k=1}^{n} f_k(y_n)x_k\right) \\
&= -f_j\left(\sum_{n=1}^{\infty} g_n(z)y_n\right) + \sum_{\{n:r_n \geq j\}} g_n(z)f_j(y_n) \\
&= -f_j\left(\sum_{\{n:r_n<j\}} g_n(z)y_n\right) \text{ by biorthogonality of } \{x_n; f_n\}:
\end{aligned}$$

So, $S^0 f_j - f_j = -\sum_{\{n:r_n<j\}} f_j(y_n)g_n \in [g_n]$ for all j: By hypothesis, $[f_n] \subseteq [g_n]$. Thus, $S^0 f_j \in [g_n]$ for all j, and we have $[h_n] \subseteq [g_n]$: ¤

4. Characterization of Sectional Boundedness

The main theorem in this section characterizes sectional boundedness of biorthogonal systems in terms of the weak¤ closure of the ball of the closed span of its functionals. This characterization will be employed to prove the invariance property of (W). Before this can be accomplished, several properties which follow from sectional boundedness of a biorthogonal system in a Banach space are necessary.

LEMMA 4.1. If $\{x_n; f_n\}$ is AB in the Banach space X, then $\{x_n\}$ is basic in X.

Proof. De¯ne $g_n : X \to X$ by $g_n(x) = x - \sum_{k=1}^{n} f_k(x)x_k$. Since $\{x_n; f_n\}$ is AB, $\{g_n\}$ is pointwise bounded, thus uniformly bounded on X. Therefore, the map $u : X \to \ell_1$; given by $ux = \{g_k(x)\}$; is continuous. Since c_0 is closed in ℓ_1; $u^{-1}(c_0) = \{x : g_n(x) \to 0\}$ is closed in X. Now, $g_n(x_k) = 0$ for $n \geq k$. Therefore, for each positive integer k, $x_k \in \{x : g_n(x) \to 0\}$, which is closed in X. Thus, $g_n \to 0$ on $[x_k]$. Equivalently, $\{x_k\}$ a basis for $[x_k]$, hence basic in X. ¤

LEMMA 4.2. Let X be a Banach space with biorthogonal system $\{x_n; f_n\}$. Then $\{x_n; f_n\}$ is AB in X i® $\{f_n; x_n\}$ is AB in X^0.

Proof. For each $f \in X^0$ and any positive integer n, we have

$$(P_n^0 f)x = f(P_n x) = \sum_{k=1}^{\infty} f(x_k)f_k(x) = \left(\sum_{k=1}^{\infty} f(x_k)f_k\right)(x):$$

As noted in Section 2, $\{x_n; f_n\}$ is AB i® $\{P_n\}$ is bounded. Also, $\{P_n^0\}$ is bounded i® $\{P_n\}$ is bounded. Lastly, $\{f_n; x_n\}$ is AB in X^0 i® $\{P_n^0\}$ is bounded as shown by the equation above. ¤

LEMMA 4.3. If $\{x_n; f_n\}$ is AB in the Banach space X, then $\{f_n\}$ is basic in X^0.

Proof. By Lemma 4.2, $\{f_n; x_n\}$ AB in X^0. Therefore, by Lemma 4.1, $\{f_n\}$ is basic in X^0. ¤

The following characterization of basic sequences can be found in [2].

PROPOSITION 4.4. $\{x_n\}$, with $x_n \neq 0$ for all n, is a basic sequence in the Banach space X iff there exists $L > 0$ such that for all $\{\alpha_k\} \in \ell$ and for all $m < n$;

$$\left\| \sum_{k=1}^{m} \alpha_k x_k \right\| \leq L \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|.$$

THEOREM 4.5. Let X be a Banach space with biorthogonal system $\{x_n; f_n\}$. Then $\{x_n; f_n\}$ is AB in X iff $\{f_n\}$ is basic in X^0 and there exists an $\varepsilon > 0$ such that $D_{X_0^0}(0;\varepsilon) \subseteq \overline{B_{[f_n]}|_{X_0}}^{w^* \text{ in } X_0^0}$.

Proof. ($\Rightarrow$) By Lemma 4.3, $\{f_n\}$ is basic in X^0. Since $\{x_n; f_n\}$ is AB we have $\sup_n \|P_n\| < \infty$. Let $D = \left\{ f \in X^0 : \|f\|_{X^0} \leq \frac{1}{\sup_n \|P_n\|} \right\}$.

Let $i^0 : X^0 \to X_0^0$ be the adjoint of the inclusion map of X_0 into X. This is the restriction mapping given by $i^0 f = f|_{X_0}$ for all $f \in X^0$. It will be shown that $i^0(D) \subseteq \overline{B_{[f_n]}|_{X_0}}^{w^* \text{ in } X_0^0}$. This will imply the existence of an $\varepsilon > 0$ such that $D_{X_0^0}(0;\varepsilon) \subseteq i^0(D)$.

Let $f \in D$. For any positive integer n, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{n} f(x_k) f_k \right\|_{X^0} &= \sup \left\{ \left| f\left(\sum_{k=1}^{n} f_k(x) x_k \right) \right| : \|x\| \leq 1 \right\} \\
&\leq \|f\| \sup \left\{ \left\| \sum_{k=1}^{n} f_k(x) x_k \right\| : \|x\| \leq 1 \right\} \\
&\leq \frac{1}{\sup_k \|P_k\|} \|P_n\| \\
&\leq 1.
\end{aligned}$$

By Lemma 4.1, $\{x_n\}$ is a basis for X_0. Therefore, the functionals restricted to X_0, $\{f_n|_{X_0}\}$, form a weak* basis for X_0^0. Thus,

$$\sum_{k=1}^{\infty} f(x_k) f_k(x) \to f(x) \text{ for all } x \in X_0;$$

that is,

$$\left(\sum_{k=1}^{n} f(x_k) f_k \right)|_{X_0} \xrightarrow{w^*} f|_{X_0} \text{ in } X_0^0.$$

Since restriction maps do not increase norms we have

$$f|_{X_0} = i^0 f \in \overline{B_{[f_n]}|_{X_0}}^{w^*\text{ in } X_0^0} \text{ for all } f \in D:$$

Since f was an arbitrary element in D, we have

$$i^0(D) \subset \overline{B_{[f_n]}|_{X_0}}^{w^*\text{ in } X_0^0}:$$

By the Hahn-Banach theorem, we have

$$\left\{ f \in X_0^0 : \|f\|_{X_0^0} \le \frac{1}{\sup_n \|P_n\|} \right\} \subset i^0(D):$$

Therefore, taking $\delta = \frac{1}{\sup_n \|P_n\|}$ satisfies this direction of the theorem.

( ) By Lemma 4.2, it is equivalent to show $\{f_n; x_n\}$ AB in X^0. Let $\delta > 0$ be given so that $D_{X_0^0}(0;\delta) \subset \overline{B_{[f_n]}|_{X_0}}^{w^*\text{ in } X_0^0}$. Let $f \in X^0$; there exists $\alpha > 0$ such that $\|\alpha f\|_{X^0} \le \delta$. Hence, $\|\alpha f|_{X_0}\|_{X_0^0} \le \delta$. Since X_0 is separable, by the Banach-Alaoglu theorem the closed sphere $D_{X_0^0}(0;\delta)$ is w^*-metrizable. Therefore, there exists a sequence $\{g_n\} \subset B_{[f_n]}$ such that

$$g_n|_{X_0} \xrightarrow{w^*} \alpha f|_{X_0} \text{ in } X_0^0:$$

So, $g_n(x) \to \alpha f(x)$ for all $x \in X_0$. In particular,

$$\lim_{n\to\infty} g_n(x_k) = \alpha f(x_k) \text{ for all positive integers } k: \tag{1}$$

Since $g_n \in B_{[f_n]}$ and $\{f_k\}$ is basic in X^0 we have

$$g_n = \sum_{k=1}^{\infty} g_n(x_k) f_k \text{ in } X^0: \tag{2}$$

Thus, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} f(x_k) f_k \right\|_{X^0} &= \left\| \sum_{k=1}^{\infty} \left(\frac{1}{\alpha} \lim_{m\to\infty} g_m(x_k) \right) f_k \right\|_{X^0} \quad \text{by (1)} \\
&\le \lim_{m\to\infty} \frac{L}{\alpha} \left\| \sum_{k=1}^{\infty} g_m(x_k) f_k \right\|_{X^0} \quad \text{by Prop. 4.4} \\
&\qquad \text{where } L \text{ is a basis constant for } \{f_k\} \\
&= \lim_{m\to\infty} \frac{L}{\alpha} \|g_m\|_{X^0} \quad \text{by (2)} \\
&\le \frac{L}{\alpha} \cdot 1 \quad \text{since } g_m \in B_{[f_n]}:
\end{aligned}$$

Therefore, $\left\{ \sum_{k=1}^{\infty} f(x_k) f_k \right\}$ is bounded in X^0. Since f was an arbitrary member of X^0, $\{f_n; x_n\}$ is AB in X^0. ∎

5. Invariance of the Wilansky Property

Recall that a basis $\{z_n\}$ for a Banach space is defined to have (W) if $\dim(Z/X) < 1$ whenever X is a BZ space with the property that Z is the smallest BZ space containing X in which $\{z_n\}$ is basic. The main theorem in this section shows that (W) as defined for a basis is an invariant of the closed span of the coefficient functionals. In order to work towards a proof of this invariance property, some results from [10] are necessary.

[10, Lemma 3.2] established the connection between the new basis definition of (W) and the old definition of (W) for BK AK spaces as follows.

PROPOSITION 5.1. For Z a BK space with basis $\{\pi^n\}$ and Y a BZ AD space, the following are equivalent:

1) Z is the smallest BZ space containing Y in which $\{\pi^n\}$ is basic.
2) $Y^{-} = Z^{-}$:

Notice that condition 2) is equivalent to $Y^{\circ} = Z^{\circ}$ since both Y and Z are AD.

Also, [10, Theorem 3.5] gives equivalent conditions for satisfying (W) in a BK AK space.

PROPOSITION 5.2. For $\{\pi^n\}$ a basis for a Banach space Z, the following conditions are equivalent:

1): $\{\pi^n\}$ has (W).
2): $Y = Z$ whenever Y is a BZ space, $Y = [\pi^n]_Y$; and with the property that Z is the smallest BZ space containing Y in which $\{\pi^n\}$ is basic.

An important contribution of this proposition is the sufficiency of considering only BZ spaces Y which are AD

Some additional definitions, facts and technical lemmas are necessary for the main theorem. Let $\{x_n\}$ be a basis for the Banach space X. Let $\{p_n\}$ and $\{q_n\}$ be intertwining sequences of positive integers such that $p_1 \cdot q_1 < p_2 \cdot q_2 < p_3 \cdot q_3 < \cdots$ Let $\alpha \in \ell^1$ and $z_n = \sum_{k=p_n}^{q_n} \alpha_k x_k$: $\{z_n\}$ is a block basic sequence with respect to $\{x_n\}$ if $z_n \neq 0$ for each n. A closed subspace M of a Banach space X is complemented in X if there exists a closed subspace N of X such that $M + N = X$ and $M \cap N = \{0\}$:

FACT 5.3. The closed subspace M is complemented in the Banach space X if there exists a continuous projection $P : X \to X$ such that $R(P) = M$:

The following proposition appeared as [10, Lemma 4.1]. It gives conditions under which a copy of ℓ_1's basis will exist in a BK AK space.

PROPOSITION 5.4. If Z is a BK AK space, X BZ AD with $X^\circ = Z^\circ$ and Z_0° is not closed in X^f, then there exists a block basic sequence, $\{b^n\}$, equivalent to ℓ_1's basis in Z such that $[b^n]_Z$ is complemented in Z.

As shown by the following lemma, every basis for a Banach space fails (W) in the presence of such a copy of ℓ_1 as produced in Proposition 5.4. A more general result based on the theory of strictly cosingular operators appeared in [10]. This lemma produces the result without reference to these operators.

LEMMA 5.5. If $\{z_n\}$ is a basis for the Banach space Z and $\{b_n\}$ is a block basic sequence equivalent to ℓ_1's basis in Z with $[b_n]_Z$ complemented in Z, then $\{z_n\}$ fails (W).

Proof. [Note: The beginning of the proof follows [10, Example 3.1].] By Fact 5.3, let P be a bounded projection of Z onto $[b_n]_Z$. So, $Z = [b_n]_Z + N(P)$. Let $\{g_n\} \subset [b_n]_Z^0$ be the coefficient functionals of $\{b_n\}$. Let

$$Y = \{y \in Z : g_{2n-1}(Py) = g_{2n}(Py) \text{ for all integers } n \geq 1\}.$$

So, $Y = \bigcap_n \ker\left((g_{2n-1} - g_{2n}) \circ P\right)$, and is therefore BZ and closed in Z. Let X be BZ with $Y \subset X \subset Z$ and $\{z_n\}$ basic in X. Once it is shown that $X = Z$ and $\dim(Z/Y) = 1$, then $\{z_n\}$ fails (W) by definition.

Let q be the quotient map, $q : Z \to Z/Y$ and consider the sequence $\{q(b_{2n-1})\}$. To show Y has infinite codimension in Z, it suffices to show that this sequence is linearly independent in Z/Y. If $\sum_{k=1}^{n} \alpha_k q(b_{2k-1}) = 0$, then $q\left(\sum_{k=1}^{n} \alpha_k b_{2k-1}\right) = 0$ by linearity of q, and $\sum_{k=1}^{n} \alpha_k b_{2k-1} \in Y$. By the definition of Y, we have $\alpha_k = 0$ for all $1 \leq k \leq n$. So, $\{q(b_{2n-1})\}$ is linearly independent in Z/Y and we have $\dim(Z/Y) = 1$.

Since $Z = [b_n]_Z + N(P)$, to show $X = Z$ it suffices to show that both $[b_n]_Z \subset X$ and $N(P) \subset X$. Certainly $N(P) \subset Y$ since $(g_{2n-1} - g_{2n})(0) = 0$. So, $N(P) \subset Y \subset X$.

Since $\{z_n\}$ is basic in X, by Proposition 4.4 there exists an $M > 0$ such that for all $\{\alpha_k\} \in \ell$ and for integers $m \leq n$,

$$\left\| \sum_{k=1}^{m} \alpha_k z_k \right\|_X \leq M \cdot \left\| \sum_{k=1}^{n} \alpha_k z_k \right\|_X .$$

Since $\{b_n\}$ is equivalent to the unit vector basis of ℓ_1 in Z, $\{b_{2n-1} + b_{2n}\}$ is bounded in Z. Also, notice that $b_{2n-1} + b_{2n} \in Y$ for all positive integers n. Since Y is closed in Z, $\{b_{2n-1} + b_{2n}\}$ is bounded in Y Since $Y \subset X$ and both are BZ, $\{b_{2n-1} + b_{2n}\}$ is bounded in X. Therefore, $\|b_{2n-1}\|_X \leq M \cdot \|b_{2n-1} + b_{2n}\|_X$ since $\{b_n\}$ is block basic. Therefore, $\{b_n\}$ is bounded in X.

Now, since $\{b_n\}$ is equivalent to the unit vector basis of ℓ_1 in Z, then every $z \in [b_n]_Z$ can be represented as $z = \sum \lambda_k b_k$, where $\lambda \in \ell_1$. With $\lambda \in \ell_1$ and $\{b_n\}$ bounded in X, $\sum \lambda_k b_k$ is absolutely convergent, hence convergent, in X. Therefore, $[b_n]_Z \subset X$.

With these two conditions satisfied, $\{z_n\}$ fails (W). ¤

LEMMA 5.6. If Z is a BK AK space, and X is BZ AD with Z°_0 closed in X^f, then $\{\varphi_n\}$ is basic in X^0.

Proof. Since all $^\circ$-duals are AB, Z° is AB, thus making Z°_0 both AB and AD, hence AK. So, $\{\pm^n\}$ is a basis, hence basic, in Z°_0. Since Z°_0 is closed in X^f; $\{\pm^n\}$ is basic in X^f. Since X is AD, then as noted in Section 2, X^f is isomorphic to X^0 with the isomorphism mapping φ_n to $\pm^n$. Since isomorphisms preserve basic sequences, $\{\varphi_n\}$ is basic in X^0. ¤

LEMMA 5.7. If Z is a Banach space with $\{f_n\} \subset Z^0$; $\{g_n\} \subset Z^0$ and $[f_n]_{Z^0} \subset [g_n]_{Z^0}$; then $[f_n|_X]_{X^0} \subset [g_n|_X]_{X^0}$ for any BZ space X.

Proof. Let X be a BZ space. Since the inclusion map $i : X \to Z$ is continuous, the corresponding adjoint mapping, $i^0 : Z^0 \to X^0$, is continuous. Notice that $i^0(f) = f \circ i = f|_X$ for $f \in Z^0$: Call this restriction mapping R, where $Rf = f|_X$. Let $f_n \in [g_k]_{Z^0}$. Then $Rf_n \in R[g_k]_{Z^0} \subset [Rg_k]_{X^0}$ since R is continuous. Thus, $[Rf_n]_{X^0} \subset [Rg_n]_{X^0}$. ¤

THEOREM 5.8. Let $\{x_n\}$ and $\{y_n\}$ be bases for the Banach space Z with coefficient functionals $\{f_n\}$ and $\{g_n\}$, respectively. If $\{x_n\}$ has (W) and $[f_n]_{Z^0} \subset [g_n]_{Z^0}$, then $\{y_n\}$ has (W).

Proof. Since $\{y_n\}$ is a basis for Z, Z has a representation as a BK AK space under the isomorphism that pairs each y_n with $\pm^n$ and each g_n with φ_n. Under this isomorphism, let each x_n be paired with z^n, and each f_n with t_n. Since (W) is invariant under equivalent bases, it is equivalent to assume that $\{z^n\}$ is a basis with (W) for this BK

AK space and $[t_n] \subseteq [\varphi_n]$: By Theorem 3.2, we can assume that $\pm^n \in \operatorname{span}\{z^k\}$ for each n. In order to show $\{y_n\}$ has (W) in Z, it is necessary to show that $\{\pm^n\}$ has (W) in this BK AK representation of Z. For simplicity of notation, the sequence space representation of Z will also be called Z.

We will assume that $\{\pm^n\}$ fails (W) and work towards a contradiction. By Proposition 5.2, choose $Y \subseteq BZ$, $[\pm^n]_Y = Y$, with $Y \subsetneq Z$ such that Z is the smallest space containing Y in which $\{\pm^n\}$ is basic. Notice that, since Y is dense in Z and $Y \subsetneq Z$, Y must have infinite codimension in Z. To reach a contradiction, we will produce a BZ space A such that $Y \subseteq A \subseteq Z$ and $\{\pm^n\}$ is basic in A.

Since $\{z^n\}$ has (W) and $\dim(Z/Y) = 1$, then by definition there exists an X such that $Y \subseteq X \subseteq Z$ and $\{z^n\}$ is basic in X. So, $\{z^n\}$ is a basis for $[z^n]_X$.

Since Y is AD and $Y \subseteq X$, we have $[\pm^n]_Y \subseteq [\pm^n]_X$: Since $\pm^n \in \operatorname{span}\{z^k\}$ for each n, we have $[\pm^n]_X \subseteq [z^n]_X$: Consequently, we have

$$Y = [\pm^n]_Y \subseteq [\pm^n]_X \subseteq [z^n]_X \subseteq X \subseteq Z:$$

The space $[\pm^n]_X$ will produce the contradiction. In order to show that $\{\pm^n\}$ is basic in $[\pm^n]_X$, we will show that $\{\pm^n; \varphi_n\}$ is AB in $[\pm^n]_X$. Then by Lemma 4.1, we will have $\{\pm^n\}$ is basic in $[\pm^n]_X$:

For convenience, let $A = [\pm^n]_X$ and $C = [z^n]_X$: In another effort to minimize notation, $[t_n]_{C^0}$, for example, will be written instead of $[t_n|_C]_{C^0}$, with the appropriate restriction of the functional implied by the subspace subscript.

By Theorem 4.5, showing that $\{\pm^n; \varphi_n\}$ is AB in A is equivalent to showing that:

(a) there exists $\varepsilon > 0$ such that $D_{A^0}(0;\varepsilon) \subseteq \overline{B}^{w^*\text{ in } A^0}_{[\varphi_n]_{A^0}}$

and

(b) $\{\varphi_n\}$ is basic in A^0:

First, we will show (a). Since $\{z^n\}$ is a basis for C, $\{z^n; t_n\}$ is AB in C. By Theorem 4.5, there exists an $\varepsilon > 0$ such that

$$D_{C^0}(0;\varepsilon) \subseteq \overline{B}^{w^*\text{ in } C^0}_{[t_n]_{C^0}}: \tag{3}$$

Let $f \in A^0$ with $\|f\|_{A^0} < \varepsilon$. Since $A \subseteq C$, by the Hahn-Banach theorem extend f to $F \in C^0$ without increase of norm. So, $f \prec F$ on A and

$$\|f\|_{A^0} = \|F\|_{C^0} < \varepsilon: \tag{4}$$

Since C is separable, by the Banach-Alaoglu theorem the closed sphere $D_{C^0}(0;\varepsilon)$ is w^*-metrizable. By (3) and (4) there exists $\{h_n\} \subseteq B_{[t_n]_{C^0}}$

such that $h_n \xrightarrow{w^*} F$ in C^0. Since $A \subset C$, then $h_n \xrightarrow{w^*} F$ in A^0. Since $F = f$ on A, we have

$$h_n \xrightarrow{w^*} f \text{ in } A^0. \tag{5}$$

By Lemma 5.7, since $[t_n]_{Z^0} \subset [\frac{1}{4}n]_{Z^0}$, we have

$$[t_n]_{C^0} \subset [\tfrac{1}{4}n]_{C^0}.$$

Therefore,

$$h_n \in B_{[t_n]_{C^0}} \subset B_{[\frac{1}{4}n]_{C^0}}.$$

Now, taking the continuous restriction mapping $R : C^0 \to A^0$, since $h_n \in [\frac{1}{4}n]_{C^0}$,

$$Rh_n \in R\left([\tfrac{1}{4}n]_{C^0}\right) \subset [R\tfrac{1}{4}n]_{A^0};$$

i.e. $h_n \in [\frac{1}{4}n]_{A^0}$. Also, $\|h_n\|_{A^0} \le \|h_n\|_{C^0} \le 1$; so we have $h_n \in B_{[\frac{1}{4}n]_{A^0}}$. By (4) and (5), the δ given above is such that

$$D_{A^0}(0;\delta) \subset \overline{B}^{w^* \text{ in } A^0}_{[\frac{1}{4}n]_{A^0}}.$$

Lastly, we will show (b). It is necessary to show that $\{\frac{1}{4}n\}$ is basic in A^0. Since $Y \subset A \subset Z$, we have $Z^\circ \subset A^\circ \subset Y^\circ$. Also, Z is the smallest space containing Y in which $\{\pm^n\}$ is basic. Then by Proposition 5.1 and the note following it, $Z^\circ = Y^\circ$. Therefore, $A^\circ = Z^\circ$.

Claim: Z°_0 is closed in A^f.

Suppose Z°_0 is not closed in A^f. Then, by Proposition 5.4, there exists a block basic sequence, $\{b^n\}$, equivalent to ℓ_1's basis in Z such that $[b^n]_Z$ is complemented in Z. By Lemma 5.5, in the presence of such a copy of ℓ_1's basis, every basis for Z fails (W). Since $\{z^n\}$ has (W), we have a contradiction. Thus, the claim has been shown.

Therefore, Z°_0 is closed in A^f and by Lemma 5.6, $\{\frac{1}{4}n\}$ is basic in A^0. ∎

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